

**Alternative algebras with invertible derivations.<sup>1</sup>****Ivan Kaygorodov, Yury Popov**Sobolev Inst. of Math., Novosibirsk, Russia,  
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## 1. INTRODUCTION.

The notion of invertible derivation as a derivation which takes only invertible or zero values appeared in work [1]. Bergen, Herstein and Lanski determined the structure of associative rings that admit invertible derivations. Later, the results of this paper were generalized in [2, 3, 4, 5, 6].

Another natural definition of invertible derivation as an invertible mapping originates from work [8], in which the relation between nilpotency of Lie algebra and invertible derivations was established. Later, the results of this paper were generalized in [9, 10, 11].

Nowadays, a great interest is shown to the studying of nearly associative algebras and superalgebras with derivations. For example, works [12, 13] determine the structure of differentially simple alternative and Jordan algebras, and papers [14, 15, 16, 17] give the description of generalizations of derivations of simple and semisimple Jordan and structurable (super)algebras. Nevertheless, the problem of specification of algebras from classical nonassociative varieties (such as alternative, Jordan, structurable, etc.), admitting invertible derivations, remains unconsidered. The present work is to make up this gap.

## 2. ALTERNATIVE ALGEBRAS WITH INVERTIBLE DERIVATIONS.

An algebra  $A$  is called *alternative* (see [18] for more information on alternative algebras), if  $A$  satisfies the following identities:

$$x^2y = x(xy), xy^2 = (xy)y.$$

It's easy to check that in any alternative algebra the associator is skew-symmetric function of its arguments, and *flexible identity*  $x(yx) = (xy)x$  holds. Also [18, p. 35] every alternative algebra satisfies the *middle Moufang identity*:

$$(xy)(zx) = x(yz)x.$$

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Let  $A$  be an algebra with unit element 1 over field  $F$ . We will denote by  $U$  the set of invertible elements of  $A$ . Further in this paragraph we will only consider invertible derivations, by which we understand such derivations  $0 \neq d$  that  $d(x) \in U$  or  $d(x) = 0$  for every  $x \in A$ .

We are using standard notations:

$$(x, y, z) := (xy)z - x(yz) \text{ — the associator of elements } x, y, z,$$

$$[x, y] := xy - yx \text{ — the commutator of elements } x, y,$$

$$x \circ y := xy + yx \text{ — the Jordan product of elements } x, y.$$

The *nucleus* of an algebra  $A$  is the set

$$N(A) = \{n \in A \mid (n, A, A) = (A, n, A) = (A, A, n) = (0)\},$$

*commutative centre* of  $A$  is the set

$$K(A) = \{k \in A \mid [k, A] = [A, k] = (0)\},$$

and *center* is  $Z(A) = N(A) \cap K(A)$ . *Centralizer* of a subset  $S \subset A$  is the set

$$C_A(S) = \{x \in A \mid [x, S] = (0)\}.$$

Derivation  $d$  is called *inner* if it lies in the smallest subspace of space of all linear operators on  $A$ , containing all right and left multiplications and closed under commutation. Otherwise  $d$  is called *outer*. The following statement is well known:

**Theorem 1.** *Let  $A$  be a simple alternative algebra that is not associative. Then the center of the algebra  $A$  is a field, and  $A$  is a Cayley–Dickson algebra over its center.*

Also,  $C$  is *quadratic* over  $F$ , that is, for every  $x \in C$  the following relation holds:

$$x^2 - t(x)x + n(x) = 0, \tag{1}$$

where  $t(x), n(x) \in F$ ,  $t(x)$  is an  $F$ –linear mapping, and  $n(x)$  is a strictly nondegenerate quadratic form satisfying  $n(xy) = n(x)n(y)$  for all  $x, y \in C$ .

In case when  $\text{char } F \neq 2$ ,  $C$  can be obtained from  $F$  by applying the Cayley–Dickson process thrice to  $F$  with the identical involution and parameters  $\alpha, \beta, \gamma \in F$ . Cayley–Dickson algebra containing zero divisors is called *split*. It's known [18] that element  $x$  of split Cayley–Dickson algebra is invertible if and only if  $n(x) \neq 0$ . It's also known [18] that every split Cayley–Dickson algebra over field  $F$  is isomorphic to *Cayley–Dickson matrix algebra*  $C(F)$ , comprising matrices of the form  $a = \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$ , where  $\alpha, \beta \in F$ ,  $u, v \in F^3$ .

Addition and scalar multiplication of elements of the algebra  $C(F)$  will then correspond to the usual addition and scalar multiplication of matrices. However, multiplication of elements of the algebra  $C(F)$  will correspond to the following matrix multiplication:

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & t \\ w & \delta \end{pmatrix} = \begin{pmatrix} \alpha\beta + (u, w) & \alpha t + \delta u - v \times w \\ \gamma v + \beta w + u \times t & \beta\gamma + (v, t) \end{pmatrix},$$

where for vectors  $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in F^3$  by

$$(x, y) = x_1y_1 + x_2y_2 + x_3y_3$$

we denote their dot product, and by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

— their cross product. Under given representation  $t(a) = \alpha + \beta$ ,  $n(a) = \alpha\beta - (u, v)$ . Further lemmas were proved in [1] for associative algebras and can be easily generalized to alternative case with minor differencies, but in order to ensure the completeness of the narration we shall provide their proofs.

**Lemma 2.** *If  $d(x) = 0$ , then either  $x = 0$ , or  $x$  is invertible.*

**Proof.** Let's notice (see, [18, p. 204]), that in every alternative algebra the following identity holds:

$$(a^{-1}, a, b) = 0. \quad (2)$$

It's easy to see that in arbitrary alternative algebra the product of two invertible elements is also invertible. Using identity (2), for invertible  $a$  and  $b$  we can find

$$\begin{aligned} (b^{-1}a^{-1})(ab) &= a^{-1}((ab)b^{-1}) - (a^{-1}(ab))b^{-1} + (b^{-1}a^{-1})(ab) = \\ &= -(a^{-1}, ab, b^{-1}) + (b^{-1}a^{-1})(ab) = -(b^{-1}, a^{-1}, ab) + (b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}(ab)) = 1. \end{aligned}$$

Assume that  $x \neq 0$ ; since  $d \neq 0$ , there exists  $y \in A$  such that  $d(y) \in U$ . Hence  $d(yx) = d(y)x \in U$  and  $d(y)^{-1}d(yx) = x$ . In view of  $d(y)$  and  $d(yx)$  being invertible,  $x$  is also invertible. The lemma is proved.

**Lemma 3.** *If  $L \neq 0$  is a one-sided ideal in  $A$  then  $d(L) \neq (0)$ .*

**Proof.** Since lemma is obvious when  $L = A$ , we should only consider the case when  $L \neq A$ . If  $0 \neq a \in L$ , then, by lemma 2,  $d(a) \neq 0$ , since  $a$  is not invertible. The lemma is proved.

**Lemma 4.** *If  $I$  is a proper one-sided ideal of  $A$ , then  $I$  is both minimal and maximal.*

**Proof.** It suffices to show that every proper one-sided ideal in  $A$  is maximal. Let  $I \subset J$  be proper one-sided ideals in  $A$ . It's easy to see that  $d(I) \cap I = (0)$  and  $I \oplus d(I)$  is also a one-sided ideal in  $A$ . By lemma 3  $d(I) \neq (0)$ , hence  $d(I)$  contains invertible elements, in consequence of which  $I \oplus d(I) = A$ . For arbitrary  $j \in J$  we have:  $j = a + d(b)$ ,  $a, b \in I$ . Consequently,  $d(b) = j - a \in J \cap d(I) = (0)$ ; thus  $j = a \in I$ . The lemma is proved.

By  $Der(A)$  we will denote the set of all derivations of algebra  $A$ . Let us fix some subset  $D \subseteq Der(A)$ . The ideal  $I$  is called a  $D$ -ideal, if for all  $\partial \in D$ ,  $x \in I$   $\partial(x) \in I$  takes place. Algebra  $A$  is then called  $D$ -simple if  $A^2 \neq 0$  and  $A$  contains no proper  $D$ -ideals (for more detailed information on  $D$ -simple algebras see [12, 13] and their references).

As immediate consequence of lemma 3 we have

**Lemma 5.** *If alternative algebra  $A$  admits an invertible derivation  $d$ , it is  $d$ -simple.*

**Lemma 6.** 1) *If  $I$  is a proper ideal of  $A$  then  $I^2 = (0)$ .*  
2) *If  $\text{char} A \neq 2$  then  $A$  is simple.*

**Proof.** 1) If  $I \neq A$  is an ideal of  $A$ , then

$$d(I^2) \subset d(I)I + Id(I) \subset I,$$

consequently, by lemma 3,  $I^2 = (0)$ , since the square of every ideal in alternative algebra is an ideal [18, p.115] and  $I$  does not contain any invertible elements.

2) Let  $2A \neq 0$  and  $I \neq (0)$ . Then, by lemma 3, there exists  $b \in I$  such that  $d(b) \in U$ . Since  $b^2 = 0$ ,

$$0 = d^2(b^2) = d^2(b)b + 2d(b)^2 + bd^2(b),$$

consequently  $2d(b)^2 \in I$ .

Now, since  $d(b)$  is invertible,  $2d(b)^2 = 0$ , therefore  $2 = 0$ . We have obtained a contradiction which proves the lemma.

**Lemma 7.** *An algebra  $C$ , which is a Cayley–Dickson algebra over its center  $Z$  admits an invertible derivation  $d$  if and only if one of the following conditions hold:*

- 1)  *$C$  is obtained by means of Cayley–Dickson process from its associative division subalgebra  $B$ :  $C = B + vB, v^2 = \gamma \in Z, \gamma \neq 0$ , where  $B = \ker d$  and  $\dim_Z B = 4$ . Furthermore, in this case an arbitrary invertible derivation  $d$  is of the form  $d(a + vb) = v(bu)$ , where  $a, b \in B, u \in B$  is a fixed element with  $t(u) = 0$ .*
- 2)  *$C$  can be represented as a direct sum:  $C = B + xB$ , where  $t(x) = 0, B = \ker d, B$  is a subfield of  $C, B = B^\perp$  and  $\dim_Z B = 4$ . Furthermore, in this case an arbitrary invertible derivation  $d$  is of the form  $d(a + xb) = b$ , where  $a, b \in B$ .*

**Proof.** It's generally known (see, for example, [21]) that every derivation of  $C$  is inner. It's easy too see then that  $Z \subseteq \ker d$  and  $d$  is a  $Z$ –linear mapping. Therefore we will consider  $C$  as  $Z$ –algebra.

Let's suppose that  $A$  allows an invertible derivation  $d$ . Let's take a subspace  $V \subset C$  such that  $\dim_Z V = 4$  and  $V$  does not contain invertible elements. For example, we can take

$$V = \left\{ \begin{pmatrix} \alpha & u \\ 0 & 0 \end{pmatrix} \mid \alpha \in Z, u \in Z^3 \right\}.$$

From lemma 2 it follows that  $\dim_Z d(V) = 4$  and  $V \cap d(V) = (0)$ , hence  $C = V \oplus d(V)$ . In particular, for any  $x \in C$  there exist  $u, v \in V$  such that  $d(x) = u + d(v)$ . Consequently,  $u = d(x - v) \in V \cap d(V) = (0)$ , and, denoting  $B = \ker d$ , we have  $C = B + V$ . By lemma 2  $B$  is a division algebra, thus  $C = B \oplus V$  and  $\dim_Z B = 4$ . Combining the facts that  $B$  is simple,  $Z(C) \subseteq Z(B)$  and applying theorem 1 we have that  $B$  is an associative subalgebra in  $C$ . In  $C$  the following relation is valid: ([18, p. 39])

$$a \circ b - t(a)b - t(b)a - f(a, b) = 0. \quad (3)$$

Putting  $b = d(a)$ , we obtain

$$a \circ d(a) - t(a)d(a) - t(d(a))a - f(a, d(a)) = 0. \quad (4)$$

Applying  $d$  on (1), we have

$$a \circ d(a) - t(a)d(a) = 0. \quad (5)$$

Subtracting (4) from (5), we obtain  $t(d(a))a + f(a, d(a)) = 0$ . If  $a$  and 1 are linearly independent over  $Z$ , then we have

$$f(a, d(a)) = 0. \quad (6)$$

In case when  $a \in Z$ , then  $a \in \ker d$  and relation (6) is obvious then. Linearizing (6), we obtain  $f(a, d(b)) + f(d(a), b) = 0$ . Consequently, for arbitrary  $a \in C$   $f(d(a), B) = -f(a, d(B)) = 0$ , and so  $d(C) \subseteq B^\perp$ . We now have to study two cases: If the restriction of the form  $f$  on  $B$  is nondegenerate, then, by ([18, p.32, Th.1])  $C$  can be obtained by means of Cayley–Dickson process from  $B$ , that is  $C = B + vB$ ,  $v^2 = \gamma \neq 0$ ,  $B^\perp = vB$ . Particularly,  $d(v) = vu$  for some  $u \in B$ , therefore for arbitrary  $a, b \in B$  we have  $d(a + vb) = d(v)b = (vu)b = v(bu)$ . Substituting in ([18, p.26])

$$n(x)f(y, w) = f(xy, xw)$$

$x = v$ ,  $y = 1$ ,  $w = u$  and using (6), we obtain  $0 = f(v, vu) = n(v)t(u)$ . Since  $v^2 = \gamma \in Z$ ,  $\gamma \neq 0$ , then  $n(v) \neq 0$  and  $t(u) = 0$ . Now, let the restriction of the form  $f$  on  $B$  be degenerate. Hence there exists  $0 \neq b \in B$  such that  $f(b, B) = 0$ . Therefore  $0 = f(b, b) = 2n(b)$ . Since  $b$  is invertible, then  $n(b) \neq 0$  and we must have  $\text{char} C = 2$ .

In  $C$  the following relation holds: ([18, p.26])

$$f(x, z)f(y, w) = f(xy, zw) + f(xw, yz). \quad (7)$$

Putting in (7)  $x = b$ ,  $z = a$ ,  $y = b^{-1}c$ ,  $w = 1$ , where  $a, c \in B$ , we have  $0 = f(b, a)f(b^{-1}c, 1) = f(c, a) + f(b, ab^{-1}c)$ , and so by arbitrariness of  $a, c$  we conclude that  $f(B, B) = 0$ , that is,  $B \subseteq B^\perp$ . Now we shall show that opposite inclusion also takes place: Suppose for a moment that there exists  $x \in B^\perp$ ,  $x \notin B$ . By (2)  $\dim_Z xB = 4$  and  $A = B \oplus xB$ . Using (7), we have  $f(a, xc) = f(a \cdot 1, xc) = -f(ac, x) + f(a, x)f(1, c) = 0$  for any  $a, c \in B$ . Consequently,  $xB \subset B^\perp$  and  $C = B^\perp$ , which contradicts the nondegeneracy of the form  $f$ . We put  $x = d^{-1}(1)$ . It's obvious that  $x \notin B$  and  $C = B \oplus xB$ . (6) implies that  $0 = f(x, 1) = t(x)$ . Now there's only left to prove that  $B$  is a field. Since for any  $a, c$   $f(a, c) = n(a+b) - n(a) - n(b)$  and  $f(B, B) = 0$ , then  $n$  is a ring homomorphism from  $B$  to  $Z$ . In view of  $B$  is simple together with  $n(1) = 1$ ,  $\ker n = 0$  and  $B$  is a subfield of  $Z$ .

Conversely, suppose that condition 1) holds, that is  $C$  is obtained by means of Cayley–Dickson process from  $B$ . Let  $0 \neq u$  be an element of  $B$  such that  $t(u) = 0$ . Consider the mapping  $d : a + vb \mapsto v(bu)$ , where  $a, b \in B$ . We are to show that  $d$  is an invertible derivation. Indeed, for any  $a, b, c, e \in B$  we have

$$\begin{aligned} d(a + vb)(c + ve) + (a + vb)d(c + ve) &= \gamma(e(u + \bar{u})\bar{b}) + v((cb + \bar{a}e)u) = \\ &= \gamma(et(u)\bar{b}) + v((cb + \bar{a}e)u) = \\ &= v((cb + \bar{a}e)u) = d((ac + \gamma e\bar{b}) + v(\bar{a}e + cb)) = d((a + vb)(c + ve)). \end{aligned}$$

Also,  $n(d(a + vb)) = n(v(bu)) = n(v)n(b)n(u) = \gamma n(b)n(u) \neq 0$  if  $b \neq 0$ , since  $B$  is a division algebra. Hence  $d(a + vb)$  is invertible for any  $a \in B$ ,  $0 \neq b \in B$ . Now assume that condition 2) holds. Consider the mapping  $d : a + xb \mapsto b$ . We are to show that  $d$  is an invertible derivation. Since we have  $B = B^\perp$ , then for any  $a \in B$   $t(a) = 0$ . Combining (3) and  $\text{char} C = 2$ , we obtain

$$[x, a] = x \circ a = t(a)x + t(x)a + f(a, x) = f(a, x) \in Z, \quad (8)$$

particularly,  $d([x, a]) = 0$ . Substituting  $x$  in (1), we deduce that  $x^2 \in Z$ . Using (8), it's easy to check that for  $a, c \in B$  the following identity holds:

$$(a, c, x) = af(c, x) + f(a, x)c + f(x, ac), \quad (9)$$

consequently,  $d((a, c, x)) = 0$ . Now we will prove that  $d$  is a derivation. For arbitrary  $a, b, c, e \in B$  we have

$$d((ax+b)(cx+e)) = d((ax)(cx) + (ax)e + b(cx) + be) = d((ax)(cx)) + d((ax)e) + d(b(cx)) + d(be).$$

Consider the last two summands:

$$\begin{aligned} d((ax)e) &= d((xa)e) = d(x(ae)) = ae, \\ d(b(cx)) &= d((bc)x) = bc. \end{aligned}$$

On the other hand,

$$d(ax+b)(cx+e) + (ax+b)d(cx+e) = a(cx+e) + (ax+b)c = a(cx) + ae + (ax)c + bc.$$

Thereby we need to show that

$$d((ax)(cx)) = a(cx) + (ax)c.$$

Transforming the corresponding expressions, we have:

$$\begin{aligned} a(cx) + (ax)c &= (ac)x + (a, c, x) + a(xc) + (a, x, c) = \\ &= (ac)x + a(cx + f(c, x)) = (a, c, x) + af(c, x). \end{aligned}$$

Using the middle Moufang identity, we obtain

$$\begin{aligned} d((ax)(cx)) &= d((xa + f(a, x))cx) = d((xa)(cx)) + f(a, x)d(cx) = \\ &= d(x(ac)x) + f(a, x)c = d(x(x(ac) + f(x, ac))) + f(a, x)c = \\ &= d(x^2ac) + f(x, ac) + f(a, x)c = f(x, ac) + f(a, x)c, \end{aligned}$$

since  $x^2 \in Z$ . Equating the expressions, we will arrive at the (9) relation, which, as was shown earlier, holds identically. Therefore  $d$  is a derivation of  $C$ . The invertibility of  $d$  is obvious. Lemma is now proved.

**Example.** In work [7] an example of split Cayley–Dickson algebra  $C$  which has a subfield of dimension 4 was provided. Let's consider an imperfect field  $F$  of characteristic 2 and elements  $\alpha, \beta \in F$  such that  $\alpha, \beta, \alpha\beta$  are linearly independent over  $F^2$ . Then subalgebra  $B$  of matrix Cayley–Dickson algebra  $C(F)$ , generated by elements

$$\begin{pmatrix} 0 & (\alpha, 0, 0) \\ (1, 0, 0) & 0 \end{pmatrix}, \begin{pmatrix} 0 & (0, \beta, 0) \\ (0, 1, 0) & 0 \end{pmatrix},$$

is a subfield of  $C$ , and  $\dim_F B = 4$ .

**Lemma 8.** *If  $A$  is not simple and not associative, then  $A = C[x]/(x^2)$ , where  $C$  is a Cayley–Dickson algebra over its center  $Z(C)$ ,  $C$  is a division algebra,  $\text{char } C = 2$ ,  $d(C) = 0$ ,  $d(x) = 1 + ax$  for some fixed  $a \in Z(C)$ , and  $d$  is an outer derivation.*

**Proof.** Combining lemma 4 and lemma 6, we have  $\text{char } A = 2$ ,  $I^2 = (0)$  for any proper ideal  $I$  in  $A$  and all proper one-sided ideals in  $A$  are both minimal and maximal. Consequently, we can easily deduce that  $A$  contains unique (left, right, two-sided) ideal  $M$  and  $M^2 = 0$ . Therefore, as in the proof of lemma 4, we have  $A = M \oplus d(M)$ , particularly, for any  $a \in A$  there  $m, n \in M$  such that  $d(a) = m + d(n)$ . Hence  $m = d(a - n) \in M \cap d(A) = (0)$  and so, denoting  $C = \ker d$ , we have  $A = C + M$ . By lemma 2  $C$  is a division algebra, therefore  $A = C \oplus M$ . We define linear mappings  $\lambda : M \rightarrow C$  and  $\mu : M \rightarrow M$  by  $d(m) = \lambda(m) + \mu(m)$  for any  $m \in M$ . It's easy to notice that for any  $a \in C, b \in M$  the following holds:

$$a\mu(b) + a\lambda(b) = ad(b) = d(ab) = \mu(ab) + \lambda(ab),$$

where  $a\mu(b) \in M, \mu(ab) \in M$  and consequently  $a\lambda(b) = \lambda(ab) \in \lambda(M)$ ; similarly  $\lambda(ba) = \lambda(b)a \in \lambda(M)$ . This implies that  $\lambda(M)$  is an ideal in  $C$ . Since  $C$  is simple and  $\lambda(M) \neq \{0\}$  we derive that  $C$  is isomorphic to  $M$  as left  $C$ -module. Putting  $x = \lambda^{-1}(1)$ , we have  $A = C \oplus Cx$ . Using the fact that  $\lambda$  is a module isomorphism it's easy to see that  $[x, C] = 0$ . Considering the identity

$$3(k, x, y) = 3(y, k, x) = 3(x, y, k) = [xy, k] - x[y, k] - [x, k]y = 0,$$

satisfied for any  $k \in K(B)$ ,  $x, y \in B$  in arbitrary alternative algebra  $B$  [18, p.136] and taking into account the structure of  $A$  we deduce that  $x \in Z(A)$ . Therefore we have  $A \cong C[x]/(x^2)$ . Now theorem 1 implies that  $C$  is a Cayley–Dickson algebra over its center  $Z(C)$ .

We can write  $\mu(x) = ax$  for some  $a \in C$ . Now, since  $x \in Z(A)$  and  $\text{char} A = 2$ , for any  $c \in C$  we have:

$$0 = d(cx + xc) = c(1 + ax) + (1 + ax)c = cax + axc = (ca + ac)x.$$

Since  $C$  is a division algebra, we obtain  $ca + ac = 0$ , thus  $a \in Z(A)$ .

Finally, since every ideal of  $A$  is invariant under action of any inner derivation,  $x \in M$ , and  $d(x) \notin M$ , it is clear that  $d$  is not inner. Lemma is now proved.

**Theorem 9.** *Let  $A$  be an alternative algebra with unit element 1, admitting invertible derivation  $d$ . Then:*

1)  *$A$  is associative and one of the following conditions hold:*

- a)  *$A$  is a division algebra  $D$ ;*
- b)  *$A$  is a  $2 \times 2$  matrix algebra  $M_2(D)$  over division algebra  $D$ ;*
- c)  *$A$  is a factor-algebra of polynomial algebra  $D[x]/(x^2)$  over division algebra  $D$ , furthermore  $\text{char} D = 2$ ,  $d(D) = 0$  u  $d(x) = 1 + ax$  for some  $a$  in the center of  $D$ ,  $d$  is an outer derivation;*

2)  *$A$  is nonassociative and one of the following conditions hold:*

- a)  *$A$  is a Cayley–Dickson algebra over its center  $Z(A)$ ;*
- b)  *$A$  is a factor-algebra of polynomial algebra  $C[x]/(x^2)$  over Cayley–Dickson division algebra, furthermore  $\text{char} C = 2$ ,  $d(C) = 0$  u  $d(x) = 1 + ax$  for some  $a$  in the center of  $C$ ,  $d$  is an outer derivation;*

**Proof.** Associative case follows from [1], nonassociative case follows from lemmas 6 and 8.

### 3. A CHARACTERIZATION OF NILPOTENT ALTERNATIVE ALGEBRAS BY INVERTIBLE LEIBNIZ-DERIVATIONS.

In 1955, Jacobson [8] proved that a Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent. The problem, whether the inverse of this statement is correct, remained open until work [19], where an example of an nilpotent Lie algebra, which derivations are nilpotent (and hence, singular), was constructed. Such types of Lie algebras are called characteristically nilpotent Lie algebras.

The study of derivations of Lie algebras lead to appearance of natural generalization — pre-derivations of Lie algebras. In [9] it is proved that Jacobson's result is also true in terms of pre-derivations. Similar to the example of Dixmier

and Lister [19] several examples of nilpotent Lie algebras, whose pre-derivations are nilpotent were presented in [9], [20].

In paper [10] a generalized notion of derivations and pre-derivation of Lie algebras is defined as Leibniz-derivation of order  $k$ . Moens proved that a Lie algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. After that, Fialowski, Khudoyberdiyev and Omirov [11] showed that with the definition of Leibniz-derivation from [10] the similar result for non Lie Leibniz algebras is not true. Namely, they gave an example of non nilpotent Leibniz algebra which admits an invertible Leibniz-derivation. In order to extend the results of paper [10] for Leibniz algebras they introduced a definition of Leibniz-derivation of Leibniz algebras which agrees with Leibniz-derivation of Lie algebras case and proved that a Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. It should be noted that there exist non-nilpotent Filippov algebras with an invertible derivation (see, [22]). Also, in [23] some generalization of prederivations of associative algebras was considered.

The main purpose of this section is proved analouge of Moens's theorem for alternative algebras.

Through the section all spaces an algebras are assumed finite dimensional.

**Definition.** A Leibniz-derivation of order  $n$  for an alternative algebra  $A$  is an endomorphism  $\phi$  of that alternative algebra satisfying the identity

$$\phi(\dots(x_1x_2)\dots)x_n = \sum_{i=1}^n (\dots(x_1x_2)\dots\phi(x_i)\dots)x_n.$$

**Theorem 10.** *An alternative algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation.*

**Proof.** Let  $A$  be a finite-dimensional alternative algebra with an invertible Leibniz-derivation  $\phi$  of order  $n$  and  $\beta(A)$  be the nilpotent radical of  $A$ . Using [18], we can establish that  $A/\beta(A)$  can be represented as finite sum of its minimal ideals, where each of them is either a full matrix algebra over some division ring, or a Cayley–Dickson algebra over its center. Therefore, algebra  $A/\beta(A)$  possesses unit element 1. We will regard  $A$  as a direct sum:  $A = A^s + \beta(A)$ . Using the idea of proof from [10] we shall prove that  $\phi(\beta(A)) \subseteq \beta(A)$ . We will remark that in case when  $\phi$  is a derivation it was proved in [24].

**Step 1.** We define on vector space  $A$  the structure of  $n$ -ary algebra  $A_n$  with multiplication

$$[a_1, a_2, \dots, a_n]_n = (\dots(a_1a_2)\dots)a_n.$$

Hence  $\phi$  is a derivation of  $n$ -ary algebra  $A_n$ . We shall show that solvable radicals  $rad(A_n)$  and  $rad(A)$  of algebras  $A_n$  and  $A$  respectively coincide. It's clear that  $rad(A) \subseteq rad(A_n)$ . Consider the natural projection  $\pi : A \rightarrow A^s$ . It's easy to see that  $\pi(rad(A_n))$  is a solvable ideal in  $A^s$ : Applying  $\pi$  to the both sides of relation

$$[A, \dots, rad(A_n), \dots, A]_n \subseteq rad(A_n),$$

and using the fact that  $1 \in A^s$ , we have

$$\pi(rad(A_n))A^s + A^s\pi(rad(A_n)) \subseteq \pi(rad(A_n)).$$

Consequently,  $\pi(rad(A_n))$  is semisimple, thus  $\pi(rad(A_n)) = 0$ .



**Step 2.** We will now show that  $\phi(\beta(A)) \subseteq \beta(A)$ . Let  $\beta(A) = \tau = \tau_1 = \text{rad}(A_n)$  and  $\tau_{t+1} = [\tau_t, \tau_t, \dots, \tau_t]_n$ , then we have

$$\tau = \tau_1 \supsetneq \tau_2 \supsetneq \dots \supsetneq \tau_p \supsetneq \tau_{p+1} = 0.$$

Since the product of two ideals in alternative algebra is also an ideal, then  $\tau_t$  is an ideal in  $A_n$  for any  $t$ .

We need to show that  $\phi^i(\tau_t) \subseteq \tau$  for any  $i$ . We are using induction on  $t$ . Induction base is trivial for  $t = p + 1$ . Now let's suppose that  $\phi^i(\tau_{t+1}) \subseteq \tau$  for arbitrary  $i$ . We need to prove that  $\phi^i(\tau_t) \subseteq \tau$  holds for any  $i$ .

The set  $\tau + \phi(\tau_t)$  is a solvable ideal of  $A_n$ , since

$$[A, \dots, A, \tau + \phi(\tau_t), A, \dots, A]_n \subseteq \tau + \tau_t + \phi(\tau_t) = \tau + \phi(\tau_t)$$

and

$$[\tau + \phi(\tau_t), \dots, \tau + \phi(\tau_t)]_n \subseteq \tau + [\phi(\tau_t), \dots, \phi(\tau_t)]_n \subseteq \tau + \phi^n(\tau_{t+1}) \subseteq \tau.$$

Now we are to show that  $\tau + \phi^k(\tau_t)$  is a solvable ideal of  $A_n$  for any  $k$ . Suppose that  $\phi^i(\tau_t) \subseteq \tau$  for each  $1 \leq i < k$ . Using induction hypothesis, we have

$$[A, \dots, A, \tau + \phi^k(\tau_t), A, \dots, A]_n \subseteq \tau + \tau_t + \phi^k(\tau_t) = \tau + \phi^k(\tau_t)$$

and

$$[\tau + \phi^k(\tau_t), \dots, \tau + \phi^k(\tau_t)]_n \subseteq \tau + [\phi^k(\tau_t), \dots, \phi^k(\tau_t)]_n \subseteq \tau + \phi^{k(n+1)}(\tau_{t+1}) \subseteq \tau.$$

Therefore,  $\phi^i(\tau_t) \subseteq \tau$  and  $\phi(\tau) \subseteq \tau$ .

**Step 3.** Considering the fact that  $\phi$  is an invertible mapping, that is, it has trivial kernel, we conclude that  $\phi(A/\beta(A)) = A/\beta(A)$ , which contradicts the unitality of  $A/\beta(A)$ , since  $\phi(1) = n\phi(1)$  and  $\dim(\phi(A/\beta(A))) < \dim(A/\beta(A))$ . Contradiction obtained implies that  $A = \beta(A)$ , that is,  $A$  is nilpotent.

**Step 4.** The converse also takes place: in order to see that nilpotent alternative algebra  $A$  with nilpotency index  $s$  has an invertible Leibniz-derivation of order  $n = [\frac{s}{2}] + 1$  it suffices to consider the sum of vector spaces  $A = W + A^n$  and linear mapping  $\phi$ , defined this way:

$$\begin{aligned} \phi(x) &= x, \text{ if } x \in W, \\ \phi(x) &= nx, \text{ if } x \in A^n. \end{aligned}$$

It is easy to see that  $\phi$  is a Leibniz-derivation for  $A$  of order  $n$ . The theorem is proved.

Further on, it's easy to check that

**Remark 11.** Over fields of positive characteristics there exist nilpotent alternative algebras possessing only singular derivations.

**Proof.** Nonassociative alternative nilpotent algebras of dimension not greater than 7 were classified in [25]. Using this classification, over field of characteristic 3 we define a 7-dimensional algebra  $A$  with basis  $\{e_1, e_2, e_3, u_1, u_2, v, w\}$  by this multiplication table:

$$\begin{aligned} e_1^2 &= u_1, e^2 = \alpha u_2, e_3^2 = e_1 e_2 = e_2 e_1 = 0, e_2 e_3 = e_3 e_2 = -v, \\ e_3 e_1 &= u_2, e_1 e_3 = u_2 - \lambda v, e_1 u_1 = u_1 e_1 = v, u_2 e_1 = \lambda w, \\ e_2 u_2 &= u_2 e_2 = w, e_3 u_1 = -u_1 e_3 = \lambda w, e_1 v = v e_1 = u_1^2 = w. \end{aligned}$$

It's easy to notice that any derivation  $D$  of  $A$  satisfies  $D(w) = 0$ .

**Remark 12.** *For arbitrary alternative algebra over field of positive characteristics  $p$  the identity map is an Leibniz-derivation of order  $p + 1$ .*

**Remark 13.** *Free alternative algebra with  $n$  generators admits an invertible derivation, but is not nilpotent.*

**Proof.** It suffices to consider a derivation which acts identically on algebra generators.

**Remark 14.** *Theorem 10, Remarks 12 and 13 take place in case of associative algebras too.*

**Remark 15.** *Following the article [24] and using methods provided in proof of Theorem 10 we can show that finite-dimensional Jordan algebra admitting an invertible derivation is nilpotent.*

#### 4. ALTERNATIVE AND JORDAN ALGEBRAS WITH $QDer = End$ .

Following Leger and Luks [26], we call an additive mapping  $f$  a quasiderivation if there exists a linear map  $Q$  such that  $Q(xy) = f(x)y + xf(y)$ . Leger and Luks described all finite-dimensional Lie algebras in which an arbitrary endomorphism is a quasiderivation. They found out that such algebra is either abelian Lie algebra, two-dimensional solvable Lie algebra or three-dimensional simple Lie algebra. Later on this result was generalized to Lie superalgebras. [27]. Also, quasiderivations and generalized derivations were studied in works [4, 14, 28], and their particular cases, called  $\delta$ -derivations, were examined, for example, in [15]-[17],[29]-[36] and in other works.

Here we shall describe all finite-dimensional Jordan and alternative algebras over arbitrary field of characteristic not equal to 2, in which every endomorphism is a quasiderivation.

**Theorem 16.** *Let  $J$  be finite-dimensional Jordan algebra over field of characteristic not equal to 2 and such that  $QDer(J) = End(J)$ . Then  $J$  is either a field or  $J$  has zero multiplication.*

**Proof.** Let  $f$  be an endomorphism of algebra  $J$ , hence there exists  $Q_f$  such that

$$Q_f(xy) = f(x)y + xf(y).$$

Suppose that  $x \in Ann(J)$ , then

$$f(x)y = Q_f(xy) - xf(y) = Q_f(xy) = 0.$$

Since  $f(x)$  can be any element of algebra  $J$ , then either  $Ann(J) = 0$  or  $J$  has zero multiplication.

Suppose for a moment that there exists  $x$  such that  $x^2 = 0$ , then for any  $f \in End(J)$  we have  $2f(x)x = Q(x^2) = 0$  and by above argument  $J$  has zero multiplication. It's now easy to notice, if we suppose that  $J$  has nonzero nilpotent radical, then considering that  $J$  is a power-associative algebra we obtain that there

exists a nilpotent element of index 2. Therefore we conclude that  $J$  has trivial nilpotent radical and consequently  $J$  can be represented as a direct sum of simple Jordan algebras. Notice (see, for example [14]) that every direct summand is  $f$ -invariant, thus from now on  $J$  will be regarded as a simple unital Jordan algebra. Description of quasiderivations of simple finite-dimensional Jordan algebras (see [14]) implies that  $f$  can be represented as a sum of a scalar mapping and a derivation of algebra  $J$ . We notice that if we denote by  $R_a$  the operator of right multiplication by element  $a \in J$ , then  $R_J + \text{Der}(J)$  is a Lie algebra under commutation and is a subalgebra in  $\text{End}(J)^{(-)}$ . Furthermore,  $R_J \cap \text{Der}(J) = 0$ , since for every unital algebra  $1R_a = a$  and  $\dim(R_J) = \dim(J)$ . But  $\dim(\text{Der}(J)) \neq \dim(J)^2 - 1$  if  $\dim(J) \neq 1$ . That is, we can conclude that  $J$  is a field. Theorem is now proved.

**Theorem 17.** *Let  $A$  be a finite-dimensional alternative algebra over field of characteristic not equal to 2 and such that  $Q\text{Der}(A) = \text{End}(A)$ . Then  $A$  is either a field or  $A$  has zero multiplication.*

**Proof.** It suffices to notice that if  $f$  is a quasiderivation of algebra  $A$  then  $f$  is a quasiderivation of algebra  $A^{(+)}$ . Furthermore, if  $A$  is an alternative algebra then  $A^{(+)}$  is a Jordan algebra. Consequently, if  $A$  satisfies the condition  $Q\text{Der}(A) = \text{End}(A)$ , then  $A^{(+)}$  satisfies this condition too, therefore it is either a field or a zero multiplication algebra. Therefore we can conclude that  $A$  is either a field or an anticommutative algebra. It's generally known that alternative anticommutative algebra is nilpotent, that is, it has nontrivial annihilator  $\text{Ann}(A)$ . So, if we have  $0 \neq x \in \text{Ann}(A)$  then  $f(x)y = 0$  for any  $y \in A$  and by above argument  $A$  has zero multiplication. The Theorem is proved.

**Remark 18.** *Theorem 17 for associative algebras can be proved as consequence of result of Leger and Luks [26]. It suffices to notice that a quasiderivation  $f$  of algebra  $A$  is also a quasiderivation of algebra  $A^{(-)}$ , and to consider associative commutative algebras and associative algebras  $A$  such that  $A^{(-)}$  is either two-dimensional solvable Lie algebra or three-dimensional simple Lie algebra.*

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